On the Bose occupancy problem with randomized energy levels

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Abstract
Assuming energy states to be an independent and identically distributed positive
sequence, a randomized version of the conventional Bose occupancy problem
is investigated.

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1. Introduction

The conventional Bose occupancy problem consists of assigning indistinguishable particles on
a prescribed deterministic energy states sequence. The purpose of this work is to initiate the
study of the following related problem: assuming energy states to constitute an independent
and identically distributed sequence of positive random variables, we wish to investigate the
corresponding randomized version of the conventional Bose occupancy problem. This simple
model does not seem to have received much attention in the past, to the best of the authors'
knowledge. Under such conditions, using a grand canonical approach, a Bose–Einstein-like
condensation phenomenon is shown to take place provided the probability mass at zero of
energies is small enough (else, probability mass of energies is not 'too strongly concentrated'
at zero); we shall quantify these allegations. Next, we shall describe this phase transition
in some details and give some illustrative examples. The origin of this phenomenon seems
to result from the conjunction of two main effects: the randomness of the energy levels
with adequate distributional behavior near zero and the indistinguishability of particles to be
allocated.

When there is condensation, we shall discuss the problems of understanding the ground-
state occupancy at critical density, together with the number of distinct occupied states. Some
limit laws are obtained in some cases. Also, some aspects of ground-state occupancy in the
condensed phase will be investigated. The way the critical line separating condensed from
fluid phases depends on the full law of the energy levels sequence is elucidated.

Finally, using a description of occupancies of the ranked energy states (the sampling
formula), we shall derive the grand-canonical statistics of states with a prescribed amount

of particles. This statistical aspect concerns the fluid phase only. We report on a Poisson factorization of this distribution in a specific low temperature, low density limit.

The theory of Bose–Einstein condensation (BEC) in random systems is still quite far from satisfactory rigorous description. Traditional BEC in an ideal quantum Bose gas occurs in momentum space, when a macroscopically large number of bosons condenses onto the ground state. The fact that the one-particle spectral gap is able to produce BEC even for strongly interacting bosons has only been established quite recently (see [16]). The situation still is unclear for a random one-particle spectrum. In Lenoble and Zagrebnov [17], rigorous arguments are developed for the case when spectral randomness is due to some form of random external potential. Here, in contrast to extreme case of the gap (zero density of states near the bottom of the spectrum), (super-) exponential suppression of the density of states known as the 'Lifshitz tail', can be observed.

It has also become increasingly clear over the last decade that condensation can also take place in real space (even in one-dimensional systems) in the steady state of a broad class of non-quantum physical systems. Examples include traffic jams on a highway, island formation on growing crystals as well as many other systems (see [9]). The condensation problem has also recently been addressed in a class of one-dimensional mass transport models [10].

In this paper, the combinatorial skeleton of a condensation phenomenon is exhibited for a system which still is of a simpler nature: a random allocation problem of bosons onto a purely random energy sequence. Perhaps is it the minimal problem setting under which a condensation problem makes sense. For this peculiar abstract model, free of possible physical nature of the system, we address and solve the condensation transition; due to its intrinsic simplicity, many facets of the problem are amenable, to some extent, to analytic calculations.

Incidentally, as a corollary of the present study giving condensation criterion, assuming the energy sequence to be completely random (that is exponentially distributed with fixed given mean), the Bose condensation phenomenon fails as a result of there being no sufficiently large enough gap between the ground state and the first excited state. Similar interest and conclusions for conditions under which Bose condensation does not take place can be found in [9].

2. Randomized Bose samples

2.1. The model and some of its statistical features

Let $\mathbf{E}_n := (E_1, \ldots, E_n)$ be a system of independent and identically distributed (iid) positive energies whose law will be specified later on (at this step, we shall however further assume that E_1 has a density f_1 with support $\subseteq \mathbb{R}_+$ and that the largest value from $\mathbf{E}_n$ goes to ∞ almost surely). Let $\beta > 0$ be the inverse of temperature. We shall presently study the randomized energy version of the classical allocation model of statistical mechanics where energy levels are originally chosen deterministically. For some related aspects of the randomized problem (see [14]). The problem under study can now be formulated as follows.

Let there be k indistinguishable particles to place on the random energy levels (or states) $\mathbf{E}_n$. Conditionally on $\mathbf{E}_n$ (for quenched disorder), let $K_{n,k}(m)$ denote the occupancy of energy level m with Bose equilibrium collective law given by

$$\mathbb{P}_{\mathbf{E}_n}(K_{n,k}(m) = k_m; m = 1, \ldots, n) = \frac{1}{Z_{k,\mathbf{E}_n}(\beta)} \prod_{m=1}^{n} e^{-\beta E_m k_m}. \qquad (2.1)$$

With $[z^k] f(z)$ the coefficient of z^k in the power-series expansion of $f(z)$ the normalizing term reads

$$Z_{k,\mathbf{E}_n}(\beta) = \sum_{\substack{k_1'+\cdots+k_n'=k \\ k_1',\ldots,k_n' \geqslant 0}} \prod_{m=1}^{n} \mathrm{e}^{-\beta E_m k_m'} = [z^k] \prod_{m=1}^{n} (1 - z\,\mathrm{e}^{-\beta E_m})^{-1}. \qquad (2.2)$$

The distribution defined by equations (2.1) and (2.2) favors configurations with minimal (interaction free) energy $H_{n,k}(\mathbf{E}_n) := \sum_{m=1}^{n} E_m k_m$. Given $\mathbf{E}_n$, it is the one maximizing entropy under the assumption that the average energy is fixed and β stands for the conjugate parameter. Here and throughout, the subscript $\mathbf{E}_n$ in $\mathbb{P}_{\mathbf{E}_n}$ (or $\mathbb{E}_{\mathbf{E}_n}$) will denote conditional probability (or expectation) given $\mathbf{E}_n$.

Remarks.

(i) In equation (2.1), $k_m \in \mathbb{N}_0 := \{0,1,2,\ldots\}$, $m = 1,\ldots,n$ with no restriction but $k_1 + \cdots + k_n = k$. Imposing the additional condition that $k_m \in \{0,1\}$, $m = 1,\ldots,n$ (the Pauli exclusion principle), would lead to a Fermi–Dirac occupancy problem which we shall not further develop specifically.

(ii) The conventional Bose occupancy problem [3] can be obtained by considering an ordered deterministic energy sequence $\epsilon_{(n)} := (\epsilon_{(1)},\ldots,\epsilon_{(n)})$ where $\epsilon_{(1)} < \cdots < \epsilon_{(n)}$, instead of $\mathbf{E}_n$ used in (2.1) and (2.2). Typically, $\epsilon_{(m)} = m^\alpha$ for some $\alpha > 1$ (see [4], for example). We therefore wish to investigate the role of randomization of the energy sequence, requiring an additional averaging over disorder.

Indeed, given there are k particles, averaging equation (2.1) over disorder $\mathbf{E}_n$, the searched Bose unconditional canonical occupancy probability clearly is

$$\mathbb{P}(K_{n,k}(m) = k_m; m = 1,\ldots,n) = \mathbb{E}\mathbb{P}_{\mathbf{E}_n}(K_{n,k}(m) = k_m; m = 1,\ldots,n). \qquad (2.3)$$

As a symmetric function of the k_m, this annealed distribution is exchangeable. In particular, $\mathbb{E}(K_{n,k}(m)) = k/n$, $m = 1,\ldots,n$. Further, for instance, with $k_1 \in \{0,\ldots,k\}$ and $\mathbf{E}_{n\setminus 1} := (E_2,\ldots,E_n)$

$$\mathbb{P}_{\mathbf{E}_n}(K_{n,k}(1) = k_1) = \frac{\mathrm{e}^{-\beta E_1 k_1}}{Z_{k,\mathbf{E}_n}(\beta)} \sum_{k_2+\cdots+k_n=k-k_1} \prod_{m=2}^{n} \mathrm{e}^{-\beta E_m k_m},$$

$$= \frac{\mathrm{e}^{-\beta E_1 k_1} Z_{k-k_1,\mathbf{E}_{n\setminus 1}}(\beta)}{Z_{k,\mathbf{E}_n}(\beta)}$$

and

$$\mathbb{P}(K_{n,k}(1) = k_1) = \mathbb{E}\left[\frac{\mathrm{e}^{-\beta E_1 k_1} Z_{k-k_1,\mathbf{E}_{n\setminus 1}}(\beta)}{Z_{k,\mathbf{E}_n}(\beta)}\right]$$

is the one-dimensional marginal of $K_{n,k}(m) = k_m; m = 1,\ldots,n$. Except for very particular cases as in the example which follows, a closed-form expression of equation (2.3) seems out of reach.

Example. As a limiting example of Bose distribution, when $\beta \downarrow 0$, the joint law of $K_{n,k}(m); m = 1,\ldots,n$ tends to the uniform Bose–Einstein distribution (see [12, 13])

$$\mathbb{P}(K_{n,k}(m) = k_m; m = 1,\ldots,n) = \frac{1}{\binom{n+k-1}{k}}$$

and the one-dimensional distribution at high temperature reads

$$\mathbb{P}(K_{n,k}(1) = k_1) = \frac{\binom{n+k-k_1-2}{k-k_1}}{\binom{n+k-1}{k}}, \qquad k_1 = 0, \ldots, k.$$

Before investigating some additional statistical properties of the Bose distribution displayed in equation (2.3), let us first stress immediate consequences of the model.

Energy of a single particle. Assume first that $k = 1$. Then the sample is made of a *single* particle. From Bose law, this particle will launch on energy level number m with conditional probability $e^{-\beta E_m}/Z_{\mathbf{E}_n}(\beta)$ where $Z_{\mathbf{E}_n}(\beta) := Z_{1,\mathbf{E}_n}(\beta) = \sum_1^n e^{-\beta E_m}$. The corresponding energy attached to this single particle, say $\mathcal{E}_n$, has conditional law given by

$$\mathbb{P}_{\mathbf{E}_n}(\mathcal{E}_n = E_m) = e^{-\beta E_m}/Z_{\mathbf{E}_n}(\beta) =: V_{m,n}(\beta), \qquad m = 1, \ldots, n.$$

The random probabilities $V_{m,n}(\beta)$ satisfy $\sum_1^n V_{m,n}(\beta) \overset{\text{a.s.}}{=} 1$ and so constitute a random partition of unity. Let now $\mathbb{E}_{\mathbf{E}_n}(e^{-\lambda \mathcal{E}_n})$ be the conditional Laplace–Stieltjes transform (LST) of $\mathcal{E}_n$. Then $\mathbb{E}_{\mathbf{E}_n}(e^{-\lambda \mathcal{E}_n}) = Z_{\mathbf{E}_n}(\lambda + \beta)/Z_{\mathbf{E}_n}(\beta)$. Averaging over $\mathbf{E}_n$,

$$\mathbb{E}(e^{-\lambda \mathcal{E}_n}) := \mathbb{E}\mathbb{E}_{\mathbf{E}_n}(e^{-\lambda \mathcal{E}_n}) = \mathbb{E}\left[Z_{\mathbf{E}_n}(\lambda + \beta)/Z_{\mathbf{E}_n}(\beta) \right]$$

characterizes the distribution of the single-particle energy. In particular,

$$\mathbb{E}(\mathcal{E}_n) = \partial_\beta \mathbb{E}\left[-\log Z_{\mathbf{E}_n}(\beta) \right].$$

In this energy-biased picking procedure $\mathbf{E}_n \to \mathcal{E}_n$, states with low energy levels are clearly favored. One therefore expects $\mathcal{E}_n$ to be stochastically smaller than the typical energy level E_n from the iid sequence $\mathbf{E}_n$.

To study the Sherrington–Kirkpatrick model for spin glasses, similar questions were raised for different purposes by Derrida in the early 1980s, under the (ruled out here) hypothesis that energies were normally distributed (see [5–7]).

Total energy. Let now $\mathcal{H}_{n,k} := \sum_{m=1}^n E_m K_{n,k}(m)$ stand for total energy of the k particles system where occupancies law is as in equation (2.1); its LST is clearly given by

$$\mathbb{E}[e^{-\lambda \mathcal{H}_{n,k}}] = \mathbb{E}\left[\frac{Z_{k,\mathbf{E}_n}(\lambda + \beta)}{Z_{k,\mathbf{E}_n}(\beta)} \right], \qquad \lambda \geqslant 0.$$

Let us now show that, when the number of energy levels is fixed, the proportions of particles tend to concentrate on the ground state as the number of particles increases.

Proposition 1. *For each $m \in \{1, \ldots, n\}$, let $\mathbf{E}_{n\backslash m} := \{E_1, \ldots E_{m-1}, E_{m+1}, \ldots E_n\}$. With n fixed, as the number of particles grows, conditionally given $\mathbf{E}_n$, we have*

$$\left(\frac{K_{n,k}(m)}{k}; m = 1, \ldots, n \right) \underset{k\uparrow\infty}{\to} (V_{m,n} := \mathbf{I}_{\{E_m < \mathbf{E}_{n\backslash m}\}}; m = 1, \ldots, n) \qquad (2.4)$$

in distribution.

Proof. Let us first consider the ordered version $\mathbf{E}_{(n)}$ of the energy sequence $\mathbf{E}_n$, namely $\mathbf{E}_{(n)} := (E_{(1),n}, \ldots, E_{(n),n})$ with $E_{(1),n} < \cdots < E_{(n),n}$. Let also $\widetilde{E}_{(m),n} := E_{(m),n} - E_{(1),n} > 0$, $m = 2, \ldots, n$. Developing the product partition function

$$\prod_{m=1}^n (1 - z e^{-\beta E_m})^{-1} = \prod_{m=1}^n (1 - z e^{-\beta E_{(m),n}})^{-1}$$

appearing in equation (2.2) into a sum of rational fractions, extracting its z^k-coefficient and isolating the ground-state term, we get

$$Z_{k,\mathbf{E}_n}(\beta) = Z_{k,\mathbf{E}_{(n)}}(\beta) = e^{-\beta E_{(1),n}k}\left(\prod_{m\neq 1}\frac{1}{1-e^{-\beta\widetilde{E}_{(m),n}}} + \sum_{m=2}^{n}e^{-\beta\widetilde{E}_{(m),n}k}\prod_{l\neq m}\frac{1}{1-e^{-\beta(E_{(l),n}-E_{(m),n})}}\right).$$

Inserting this expression into equation (2.1), with $K_{(n),k}(m); m = 2,\ldots,n$ the occupancies of energy levels $E_{(2),n},\ldots,E_{(n),n}$ and $k_m \in \mathbb{N}_0; m = 2,\ldots,n$, such that $\sum_{m=2}^{n}k_m \leqslant k$, we get

$$\mathbb{P}_{\mathbf{E}_{(n)}}(K_{(n),k}(m) = k_m; m = 2,\ldots,n) = (1-\varepsilon(k))\prod_{m=2}^{n}\{e^{-\beta\widetilde{E}_{(m),n}k_m}(1-e^{-\beta\widetilde{E}_{(m),n}})\} \qquad (2.5)$$

where $\varepsilon(k) = e^{-\beta\widetilde{E}_{(2),n}k}\prod_{l\neq 1}(1-e^{-\beta\widetilde{E}_{(l),n}})/\prod_{l\neq 2}(1-e^{-\beta(E_{(l),n}-E_{(2),n})})$ is a negative corrective term that goes to 0 exponentially fast with k becoming large. Suppose $k_m = \lfloor ks_m \rfloor$ for some fixed $s_m \in (0,1]; m = 2,\ldots,n$. In this case, the probability displayed in equation (2.5) goes to 0 when k goes to ∞: in other words, the probabilities of $K_{(n),k}(m)/k; m = 2,\ldots,n$ all concentrate on 0 and therefore all the probability mass goes to ground state ($m = 1$). This is statement in equation (2.4) where $\mathbf{I}_{\{A\}}$ stands for the set indicator of the event A. Note that $\left(V_{m,n} := \mathbf{I}_{\{E_m < \mathbf{E}_{n\setminus m}\}}; m = 1,\ldots,n\right)$ is an exchangeable partition of unity. In particular, for each m, $\mathbb{E}(V_{m,n}) = 1/n$ and $\sigma^2(V_{m,n}) = (1-1/n)/n$. $\square$

Some remarks on Maxwell–Boltzmann version of the occupancy problem. This limiting behavior of occupancies is in sharp contrast with the one arising in a Maxwell–Boltzmann version of the randomized occupancy problem. Let us briefly address this point.

With $\sum_{1}^{n}V_{m,n}(\beta) \overset{\text{a.s.}}{=} 1$, the random numbers $V_{m,n}(\beta) = \frac{e^{-\beta E_m}}{\sum_{1}^{n}e^{-\beta E_m}}$ have been shown to be the conditional Bose probabilities that a single particle hits state m with energy E_m. The partitioning system $\mathbf{V}_n(\beta) := (V_{1,n}(\beta),\ldots,V_{n,n}(\beta))$ constitute an exchangeable random partition of unity (in particular, for each m, $\mathbb{E}(V_{m,n}(\beta)) = 1/n$). The random quantities $\mathbf{V}_n(\beta)$ are essential in the derivation of the Maxwell–Boltzmann sampling distribution version of the randomized occupancy problem. It proceeds as follows: let $(U_1,\ldots,U_k)$ be k iid uniform throws on $[0,1]$ partitioned by $\mathbf{V}_n(\beta)$. Let $(B_{n,k}(1),\ldots,B_{n,k}(n))$ be an integral-valued random vector which counts the number of visits to the different energy levels in a k-sample in the following sense: if M_l is the random state label which the lth trial hits, then $B_{n,k}(m) := \sum_{l=1}^{k}\mathbf{I}_{\{M_l=m\}}, m = 1,\ldots,n$. Clearly $\mathbb{P}_{\mathbf{V}_n(\beta)}(M_l = m) = V_{m,n}(\beta)$ and, given $\mathbf{V}_n(\beta)$, state m is chosen proportionally to its 'size' $V_{m,n}(\beta)$.

With $(b_1,\ldots,b_n) \in \mathbb{N}_0^n$, satisfying $\sum_{m=1}^{n}b_m = k$, $(B_{n,k}(m) = b_m; m = 1,\ldots,n)$ now follows the conditional multinomial distribution:

$$\mathbb{P}_{\mathbf{V}_n(\beta)}(B_{n,k}(m) = b_m; m = 1,\ldots,n) = \frac{k!}{\prod_{m=1}^{n}b_m!}\prod_{m=1}^{n}V_{m,n}^{b_m}(\beta).$$

Averaging over $\mathbf{V}_n(\beta)$ gives the Maxwell–Boltzmann distribution

$$\mathbb{P}(B_{n,k}(m) = b_m; m = 1,\ldots,n) = \mathbb{E}\mathbb{P}_{\mathbf{V}_n(\beta)}(B_{n,k}(m) = b_m; m = 1,\ldots,n).$$

Clearly, proceeding in this way to fill up sequentially the energy levels, particles are assumed distinguishable and, by strong law of large numbers, conditionally given $\mathbf{E}_n$

$$\left(\frac{B_{n,k}(m)}{k}; m = 1,\ldots,n\right) \underset{k\uparrow\infty}{\to} (V_{m,n}(\beta); m = 1,\ldots,n)$$

almost surely and in distribution (compare with equation (2.4)).

Note that computational problems also arise in the canonical ensemble, since, averaging over $\mathbf{E}_n$, one needs to evaluate the awkward quantities

$$\mathbb{E}\left[\prod_{m=1}^{n} V_{m,n}^{b_m}(\beta)\right] = \mathbb{E}\left[\frac{\mathrm{e}^{-\beta \sum_{m=1}^{n} E_m b_m}}{\left(\sum_{m=1}^{n} \mathrm{e}^{-\beta E_m}\right)^k}\right].$$

Due to these computational difficulties, canonical randomized Maxwell Boltzmann statistics turns out to be as intricate as Bose one. We shall not enter into more details on this question and rather focus on the Bose occupancy problem as defined by equations (2.1) and (2.2).

Random allocation scheme representation of Bose distribution. From equations (2.1)–(2.2), we observe immediately the distributional identity

$$(K_{n,k}(m) = k_m; m = 1, \ldots, n) \stackrel{d}{=} (\xi_1, \ldots, \xi_n \mid \zeta_n = k), \tag{2.6}$$

where $(\xi_1, \ldots, \xi_n)$ is iid on $\mathbb{N}_0^n$ conditioned by its sum $\zeta_n := \sum_1^n \xi_m = k$ and

$$\mathbb{P}(\xi_1 = k_1) = \mathbb{E}\big[\mathbb{P}_{\mathbf{E}_n}(\xi_1 = k_1)\big] = \mathbb{E}[\mathrm{e}^{-\beta E_1 k_1}(1 - \mathrm{e}^{-\beta E_1})], \qquad k_1 \in \mathbb{N}_0, \tag{2.7}$$

a mixture of geometric distribution with success probability $\mathrm{e}^{-\beta E_1}$. Such a conditional representation of the occupancies is called a random allocation scheme property in [15].

Let $\rho > 0$ be the density of particles per state and let $\kappa_n := \lfloor n\rho \rfloor$.

Proposition 2. *Assume $\mathbb{E}[(\mathrm{e}^{\alpha+\beta E_1} - 1)^{-1}] = \rho$ admits a solution for $\alpha := \alpha(\beta, \rho) > 0$. Then*

$$K_{n,\kappa_n}(1) \xrightarrow[n\uparrow\infty]{d} \mathcal{K}_\rho, \tag{2.8}$$

where, with $\alpha := \alpha(\beta, \rho)$, $\mathcal{K}_\rho$ is a mean-ρ mixture of geometric distribution with probability generating function

$$\mathbb{E}[u^{\mathcal{K}_\rho}] = \mathbb{E}\left[\frac{1 - \mathrm{e}^{-(\alpha+\beta E_1)}}{1 - u\,\mathrm{e}^{-(\alpha+\beta E_1)}}\right], \qquad u \in [0, 1]. \tag{2.9}$$

Proof. This follows from the above allocation representation property and the Gibbs conditioning principle. The random variable $\mathcal{K}_\rho$ is the asymptotic typical cell occupation at density ρ of particles per state. Its law can be obtained by exponential tilting those of $\xi_m, m = 1, \ldots, n$ namely, with $\mathbb{P}_{\mathbf{E}_n}(\xi_m = k_m) = \mathrm{e}^{-\beta E_m k_m}(1 - \mathrm{e}^{-\beta E_m})$, by considering

$$\mathbb{P}(\widetilde{\xi}_{m,\alpha} = k_1) := \mathbb{E}\big(\mathbb{P}_{\mathbf{E}_n}(\widetilde{\xi}_{m,\alpha} = k_m)\big) = \mathbb{E}\left[\frac{\mathrm{e}^{-\alpha k_m}\mathbb{P}_{\mathbf{E}_n}(\xi_m = k_m)}{\sum_{k_m \geqslant 0} \mathrm{e}^{-\alpha k_m}\mathbb{P}_{\mathbf{E}_n}(\xi_m = k_m)}\right].$$

Given $\mathbf{E}_n, \widetilde{\xi}_{m,\alpha}, m = 1, \ldots, n$ are therefore geometrically distributed with iid success random probabilities $\mathrm{e}^{-(\alpha+\beta E_m)}$ and expected values $(\mathrm{e}^{\alpha+\beta E_m} - 1)^{-1}$. Applying Gibbs conditioning principle to obtain equation (2.8), one needs to adjust α so as to meet the conditioning requirement $\widetilde{\zeta}_{n,\alpha} := \sum_{m=1}^{n} \widetilde{\xi}_{m,\alpha} = \lfloor n\rho \rfloor$. This will be possible at the only condition that there exists an $\alpha > 0$ such that

$$\mathbb{E}[\mathcal{K}_\rho] = \mathbb{E}[(\mathrm{e}^{\alpha+\beta E_1} - 1)^{-1}] = \rho. \tag{2.10}$$

This condition will appear more clearly later to separate, in some cases, between a fluid and a condensed phase. In case there is no α for a given ρ (see below for examples in the condensed phase), there is no typical limiting cell occupation at density ρ. $\qquad\square$

Gibbs randomization of sample size (variable particle number).   As it was stressed above, the canonical conditional distributions given sample size is k are extremely difficult to evaluate in general (even when $k = 1$). To circumvent this drawback, we shall now assume the number of particles $K_{n,\alpha}$ is now random with law given by the Gibbs probability measure

$$\mathbb{P}_{\mathbf{E}_n}(K_{n,\alpha} = k) = \frac{\mathrm{e}^{-\alpha k} Z_{k,\mathbf{E}_n}(\beta)}{Z_{\mathbf{E}_n}(\alpha, \beta)}, \tag{2.11}$$

where the grand canonical partition now reads

$$Z_{\mathbf{E}_n}(\alpha, \beta) = \sum_{k \geqslant 0} \mathrm{e}^{-\alpha k} Z_{k,\mathbf{E}_n}(\beta) = \prod_{m=1}^{n} \frac{1}{1 - \mathrm{e}^{-(\alpha + \beta E_m)}}. \tag{2.12}$$

Alternatively, with $\lambda \geqslant 0$, we have

$$\mathbb{E}_{\mathbf{E}_n}(\mathrm{e}^{-\lambda K_{n,\alpha}}) = \frac{Z_{\mathbf{E}_n}(\alpha + \lambda, \beta)}{Z_{\mathbf{E}_n}(\alpha, \beta)} = \prod_{m=1}^{n} \frac{1 - \mathrm{e}^{-(\alpha + \beta E_m)}}{1 - \mathrm{e}^{-(\lambda + \alpha + \beta E_m)}}$$

$$\mathbb{E}(\mathrm{e}^{-\lambda K_{n,\alpha}}) = \left[\mathbb{E}\left(\frac{1 - \mathrm{e}^{-(\alpha + \beta E_1)}}{1 - \mathrm{e}^{-(\lambda + \alpha + \beta E_1)}} \right) \right]^n.$$

Given $\mathbf{E}_n$, the expected number of particles can be obtained through

$$\mathbb{E}_{\mathbf{E}_n}(K_{n,\alpha}) = -\partial_\alpha \log Z_{\mathbf{E}_n}(\alpha, \beta).$$

Indexing now cell occupancies by α rather than k, with $k_m \in \mathbb{N}_0$; $m = 1, \ldots, n$, the joint occupancies probability reads

$$\mathbb{P}_{\mathbf{E}_n}\left(K_{n,\alpha}(m) = k_m; m = 1, \ldots, n\right) = \frac{1}{Z_{\mathbf{E}_n}(\alpha, \beta)} \prod_{m=1}^{n} \mathrm{e}^{-(\alpha + \beta E_m) k_m},$$

and $K_{n,\alpha} := \sum_{m=1}^{n} K_{n,\alpha}(m)$.

Let $\phi_1(\beta) := \mathbb{E}\, \mathrm{e}^{-\beta E_1}$ be the Laplace–Stieltjes transform of E_1. Averaging over $\mathbf{E}_n$, the unconditional occupancy probability is

$$\mathbb{P}(K_{n,\alpha}(m) = k_m; m = 1, \ldots, n) := \mathbb{E}\mathbb{P}_{\mathbf{E}_n}(K_{n,\alpha}(m) = k_m; m = 1, \ldots, n)$$

$$= \mathbb{E}\left[\prod_{m=1}^{n} \mathrm{e}^{-(\alpha + \beta E_m) k_m} (1 - \mathrm{e}^{-(\alpha + \beta E_m)}) \right]$$

$$= \prod_{m=1}^{n} \mathbb{E}[\mathrm{e}^{-(\alpha + \beta E_1) k_m} (1 - \mathrm{e}^{-(\alpha + \beta E_1)})]$$

$$= \prod_{m=1}^{n} (\mathrm{e}^{-\alpha k_m} [\phi_1(\beta k_m) - \mathrm{e}^{-\alpha} \phi_1(\beta(k_m + 1))]). \tag{2.13}$$

It is exchangeable and under the form of a mere product measure. In particular, with $k_1 \in \mathbb{N}_0$

$$\mathbb{P}(K_{n,\alpha}(1) = k_1) = \mathrm{e}^{-\alpha k_1} [\phi_1(\beta k_1) - \mathrm{e}^{-\alpha} \phi_1(\beta(k_1 + 1))].$$

Stated differently, with $\xi_{1,\alpha}$ the $\mathbb{N}_0$-valued random variable with distribution $\mathbb{P}(\xi_{1,\alpha} \geqslant k_1) = \mathrm{e}^{-\alpha k_1} \phi_1(\beta k_1)$, we have

$$\mathbb{P}(K_{n,\alpha}(m) = k_m; m = 1, \ldots, n) = \prod_{m=1}^{n} \mathbb{P}(\xi_{m,\alpha} = k_m),$$

where $(\xi_{1,\alpha}, \ldots, \xi_{n,\alpha})$ is an iid sequence. This factorized representation turns out to be useful.

(1) For example, let $K_{n,\alpha}^+$ be the grand-canonical occupancy of state with the largest amount of particles. Then

$$\mathbb{P}\left(K_{n,\alpha}^+ \leqslant k\right) = (1 - e^{-\alpha k}\phi_1(\beta k))^n$$

and one expects some Poisson approximation. However, following Embrechts *et al* [8, theorem 3.1.3, p 117], observing that

$$\frac{\mathbb{P}(\xi_{1,\alpha} \geqslant k_1)}{\mathbb{P}(\xi_{1,\alpha} \geqslant k_1 - 1)} \leqslant e^{-\alpha} \in (0,1) \qquad \text{and} \qquad \lim_{k_1\uparrow\infty} \frac{\mathbb{P}(\xi_{1,\alpha} \geqslant k_1)}{\mathbb{P}(\xi_{1,\alpha} \geqslant k_1 - 1)} < 1,$$

there is no sequence k_n such that $n\,e^{-\alpha k_n}\phi_1(\beta k_n) \to \gamma \in (0,\infty)$. Therefore, whatever the sequence k_n is, there is no non-degenerate limit of the form $\mathbb{P}\left(K_{n,\alpha}^+ \leqslant k_n\right) \to x$ where $x \in [0,1]\backslash\{0,1\}$; in other words, for every sequence k_n such that $\mathbb{P}\left(K_{n,\alpha}^+ \leqslant k_n\right) \to x$, either $x = 0$ or 1 and there is no way to normalize $K_{n,\alpha}^+$ so as to obtain a non-degenerate limit distribution. The discreteness of the distributions involved prevents the maximum from converging properly to some extreme value distribution and instead forces this oscillatory behavior.

(2) The grand canonical distribution of total energy: let us consider the LST of the random variable $\mathcal{H}_{n,\alpha}$. With $\lambda \geqslant 0$ and $k_m \in \mathbb{N}_0$, it is

$$\mathbb{E}(e^{-\lambda \mathcal{H}_{n,\alpha}}) = \mathbb{E}\left(\sum_{k_m; m=1,\dots,n} e^{-\lambda \sum_{m=1}^n k_m E_m}\mathbb{P}_{\mathbf{E}_n}(K_{n,\alpha}(m) = k_m; m = 1,\dots,n)\right)$$

$$= \mathbb{E}\left(\prod_{m=1}^n \sum_{k_m} e^{-(\alpha+(\lambda+\beta)E_m)k_m}(1 - e^{-(\alpha+\beta E_m)})\right)$$

$$= \mathbb{E}\left(\prod_{m=1}^n \frac{1 - e^{-(\alpha+\beta E_m)}}{1 - e^{-(\alpha+(\lambda+\beta)E_m)}}\right) = \left\{\mathbb{E}\left(\frac{1 - e^{-(\alpha+\beta E_1)}}{1 - e^{-(\alpha+(\lambda+\beta)E_1)}}\right)\right\}^n.$$

This shows that $\mathcal{H}_{n,\alpha}$ is the sum of n iid random variables whose increment has LST given by $\mathbb{E}\left(\frac{1-e^{-(\alpha+\beta E_1)}}{1-e^{-(\alpha+(\lambda+\beta)E_1)}}\right)$. In particular, by strong law of large numbers, $\mathcal{H}_{n,\alpha}/n$ converges a.s. to the mean energy per state, namely

$$\frac{\mathcal{H}_{n,\alpha}}{n} \to_{n\uparrow\infty} \mathbb{E}\left(\frac{E_1}{e^{\alpha+\beta E_1} - 1}\right).$$

2.2. A condensation phenomenon

The origin of the condensation phenomenon to be described now results from the conjunction of two main effects of the model under study: the randomness of the energy levels (the system is disordered) and the indistinguishability of particles to be allocated. Phase transitions induced by disorder is well known in related contexts (see [17]).

A Bose–Einstein-like condensation phenomenon. It remains to interpret α in more details. Observing that

$$\mathbb{E}_{\mathbf{E}_n}(K_{n,\alpha}(m)) = -\frac{1}{\beta}\partial_{E_m}\log Z_{\mathbf{E}_n}(\alpha,\beta) = \frac{1}{e^{\alpha+\beta E_m} - 1},$$

we have

$$\mathbb{E}_{\mathbf{E}_n}(K_{n,\alpha}) = \sum_{m=1}^n \frac{1}{e^{\alpha+\beta E_m} - 1} \qquad \nu := \mathbb{E}\mathbb{E}_{\mathbf{E}_n}(K_{n,\alpha}) = n\mathbb{E}((e^{\alpha+\beta E_1} - 1)^{-1}).$$

Assuming the number of levels n to be the extensive variable of such a mean-field model, we thus get the particle density per state

$$\rho := \frac{\nu}{n} = \mathbb{E}((e^{\alpha + \beta E_1} - 1)^{-1}). \qquad (2.14)$$

Usually, it is therefore of interest to try to deduce α from (ρ, β) which are the physical quantities which are known in practice. Letting $z := e^{-\alpha}$ to denote fugacity, developing the argument of the expectation, this is also the implicit state equation

$$\rho = z h_\beta(z), \qquad (2.15)$$

where $h_\beta(z)$ has the power series representation

$$h_\beta(z) = \sum_{i \geqslant 0} z^i h_i(\beta) \qquad \text{with} \qquad h_i(\beta) = \phi_1 (\beta (i + 1)). \qquad (2.16)$$

Let $z_c > 0$ be the convergence radius of the series

$$H_\beta(z) := z h_\beta(z) = \sum_{i \geqslant 1} z^i \phi_1(\beta i).$$

Two cases arise: $z_c = 1$ or $z_c > 1$. If $z_c > 1$, $h_\beta(1)$ exists, is monotone decreasing and maps $\beta \in (0, \infty)$ onto $(\infty, 0)$; let therefore β_c be defined by $\rho = h_{\beta_c}(1)$. If $z_c = 1$, let β_c be defined as before if $H_\beta(z_c) < \infty$ and $\beta_c := \infty$ if $H_\beta(z_c) = \infty$. In any case, for $\beta < \beta_c$, by Bürmann–Lagrange inversion formula, $\alpha = -\log z$ where $z \in (0, 1 \wedge z_c)$ and

$$z = 1 + \sum_{l \geqslant 1} \frac{\rho^l}{l} h_l(\beta) \qquad \text{with} \qquad h_l(\beta) := [z^{l-1}] h_\beta(z)^{-l}. \qquad (2.17)$$

A Bose–Einstein condensation-like phenomenon therefore pops in when the probability that energies take values close to 0 are small enough. We shall now supply three examples.

Examples. In example (i), the probability mass in 0 of energy is chosen log-algebraically large. In the subsequent example (ii), the probability mass of energy in 0 is exponentially small. These two examples show that a condensation-like phenomenon occurs if the probability distribution of energy in a neighborhood of 0 is small enough (with growth at least linear). In a last extreme example (iii), the support of the law of E_1 is (ϵ_0, ∞), $\epsilon_0 > 0$; ground state ϵ_0 is away from 0 with probability 1.

(i) Assume the distribution of energy E_1, say $F_1(\epsilon) := \int^\epsilon f_1(\epsilon') \, d\epsilon'$, satisfies

$$F_1(\epsilon) \sim_{\epsilon \downarrow 0} C \cdot \epsilon^\theta / L(1/\epsilon),$$

where $C > 0$, $\theta \geqslant 0$ and $L(x)$ a slowly varying at ∞ function (such as $\log x$). The smaller θ is, the more energy is concentrated at 0. Then by Karamata–Tauberian theorem (see [2])

$$\phi_1(\beta) \sim_{\beta \uparrow \infty} C \cdot \Gamma(1 + \theta) \cdot \beta^{-\theta} / L(\beta)$$

and $\phi_1(\beta)$ is regularly varying at ∞. For all $\theta \geqslant 0$, $\beta > 0$, the convergence radius of $H_\beta(z)$ is $z_c = 1$. If $\theta \leqslant 1$, $H_\beta(z_c) = \infty$ whereas for $\theta > 1$, $H_\beta(z_c) < \infty$. If $\theta \leqslant 1$, there is no condensation whereas when $\theta > 1$, a critical value β_c of β can be defined. When $\theta > 1$, the order of differentiability of $z \to H_\beta(z)$ at $z_c = 1$ is $\lfloor \theta \rfloor - 1$.
In all such cases, $\phi_1(2\beta)/\phi_1(\beta) \sim_{\beta \uparrow \infty} 2^{-\theta}$ does not tend to 0 when $\beta \uparrow \infty$.

(ii) Assume density of E_1 satisfies $f_1(\epsilon) \sim_{\epsilon\downarrow 0} C \cdot \epsilon^{-3/2} e^{-1/\epsilon}$: a small value of E_1 is rather unlikely (exponentially small). Then, $\phi_1(\beta) \sim_{\beta\uparrow\infty} C \cdot e^{-\sqrt{\beta}}$ so that $z_c = 1$ with $H_\beta(z_c) < \infty$. In this case, $\phi_1(2\beta)/\phi_1(\beta)$ does tend to 0 when $\beta \uparrow \infty$.

This example is a particular case of the related one for which $f_1(\epsilon) \sim_{\epsilon\downarrow 0} C \cdot e^{-1/\epsilon^c}$ for some $c > 0$. In this case, a small value of E_1 is exponentially–algebraically small and, by applying the saddle point method: $\phi_1(\beta) \sim_{\beta\uparrow\infty} C \cdot \beta^{-\frac{c+2}{2(c+1)}} e^{-B \cdot \beta^{c/(c+1)}}$ for some constant $B > 0$. Here again, $\phi_1(2\beta)/\phi_1(\beta) \to 0$ when $\beta \uparrow \infty$.

(iii) Let $\epsilon_0 > 0$ denote spectral gap and $a > 0$. Assume $f_1(\epsilon) = a e^{-a(\epsilon-\epsilon_0)}, \epsilon > \epsilon_0$ (a truncated exponential model). Then $\phi_1(\beta) = \frac{e^{-\beta\epsilon_0}}{1+\beta/a}$. Here $z_c = e^{\beta\epsilon_0} > 1$ and $H_\beta(z_c) = \infty$. Note that $H_\beta(z)$ is infinitely differentiable (smooth) at $z = 1 < z_c$. In this case, β_c is uniquely determined by $\sum_{i \geqslant 1} \frac{e^{-\beta_c\epsilon_0 i}}{1+(\beta_c i)/a} = \rho$.

Remark. In this approach, the free parameter is density ρ and the Lagrange inversion formula only holds for $\beta < \beta_c$ (temperature is high enough). If β is the free parameter, Lagrange inversion formula only is valid for $\rho < \rho_c := h_\beta(1)$ (the free Bose gas has small enough density). In the (β, ρ) plane, the critical line $\rho_c =: \rho_c(\beta)$ separates a diluted fluid phase $\rho < \rho_c$ from a condensed phase $\rho > \rho_c$. Note that $\rho_c(\beta) = \mathbb{E}((e^{\beta E_1} - 1)^{-1})$.

One can summarize shortly the results as follows:

Proposition 3. *We have*

(i) If $\mathbb{E}(1/E_1) = \infty$, there is no condensation and the only available phase is fluid.

(ii) If $\mathbb{E}(1/E_1) < \infty$, there is condensation: the critical line separating the condensed $(\rho > \rho_c)$ from fluid $(\rho < \rho_c)$ phases has equation $\rho = \rho_c(\beta) = \mathbb{E}((e^{\beta E_1} - 1)^{-1})$ which is also the convergent series

$$\rho = \rho_c(\beta) = \sum_{i \geqslant 1} \phi_1(\beta i)$$

involving the LST of energies.

Thermodynamics of the diluted fluid phase.   Internal energy reads $U_n(\alpha, \beta) = \mathbb{E}(U_{\mathbf{E}_n}(\alpha, \beta))$ where

$$U_{\mathbf{E}_n}(\alpha, \beta) = -\partial_\beta \log Z_{\mathbf{E}_n}(\alpha, \beta) = \sum_{m=1}^{n} \frac{E_m}{e^{\alpha+\beta E_m} - 1}.$$

Observing that $\mathbb{E}\left(\frac{E_1}{e^{\alpha+\beta E_1} - 1}\right) < \infty$ we get

$$\frac{U_{\mathbf{E}_n}(\alpha, \beta)}{n} \xrightarrow[n\uparrow\infty]{\text{a.s.}} \mathbb{E}\left(\frac{E_1}{e^{\alpha+\beta E_1} - 1}\right) =: u(\alpha, \beta).$$

Further, $U_n(\alpha, \beta) = n \cdot u(\alpha, \beta)$. The quantity $u(\alpha, \beta)$ is internal energy per state.

Assume $\mathbb{E} \log(1 - e^{-(\alpha+\beta E_1)}) < \infty$ for each $(\alpha, \beta) \in \mathbb{R}_+^2$. With $N_n(\epsilon) := \sum_{m=1}^{n} \mathbf{I}_{\{E_m \leqslant \epsilon\}}$, let the empirical distribution function (or density of states) of $\mathbf{E}_n$ be $D_n(\epsilon) := n^{-1} N_n(\epsilon)$. Let $F_{\mathbf{E}_n}(\alpha, \beta) := -\frac{1}{\beta} \log Z_{\mathbf{E}_n}(\alpha, \beta)$ be the conditional free energy. Then, by the strong law of large numbers

$$-\frac{1}{n} F_{\mathbf{E}_n}(\alpha, \beta) = \frac{1}{n\beta} \sum_{m=1}^{n} \log(1 - e^{-(\alpha+\beta E_m)})$$

$$= \frac{1}{\beta} \int_{\mathbb{R}_+} \log(1 - e^{-(\alpha+\beta\epsilon)}) D_n(d\epsilon) \xrightarrow[n\uparrow\infty]{\text{a.s.}} \frac{1}{\beta} \mathbb{E} \log(1 - e^{-(\alpha+\beta E_1)}) =: f(\alpha, \beta)$$

and

$$F_n(\alpha, \beta) := \mathbb{E}\big(F_{\mathbf{E}_n}(\alpha, \beta)\big) = \frac{n}{\beta}\mathbb{E}\log(1 - \mathrm{e}^{-(\alpha+\beta E_1)}) = n \cdot f(\alpha, \beta).$$

The quantity $f(\alpha, \beta)$ is free energy per state. We have $f(\alpha, \beta) > \frac{1}{\beta}\log(1 - \mathrm{e}^{-\alpha})$.

As usual, we can define entropy $S_n(\alpha, \beta)$ as

$$S_n(\alpha, \beta) = \beta\left(U_n(\alpha, \beta) - F_n(\alpha, \beta)\right).$$

Clearly, $S_n(\alpha, \beta) = n \cdot s(\alpha, \beta)$ where $s(\alpha, \beta) = \beta\left(u(\alpha, \beta) - f(\alpha, \beta)\right)$ is entropy per state. We have

$$s(\alpha, \beta) = \mathbb{E}\left(\frac{\beta E_1}{\mathrm{e}^{\alpha+\beta E_1} - 1}\right) - \mathbb{E}\log\left(1 - \mathrm{e}^{-(\alpha+\beta E_1)}\right).$$

Remark. Assume $n \uparrow \infty$, $\nu \uparrow \infty$ in such a way that particle density $\rho = \nu/n$ exists. Then, using the equation of state as defined by equations (2.15) and (2.16), there is an expression of α in terms of ρ and β, provided the system is in the diluted phase. The thermodynamical quantities per state $u(\alpha, \beta)$, $f(\alpha, \beta)$ and $s(\alpha, \beta)$ can better be expressed in terms of ρ and β. The functions $s(\alpha, \beta)$ and $\beta f(\alpha, \beta)$ are β-Legendre conjugates.

The existence of a limit for the thermodynamical quantities per state when the system becomes infinite is clear (see [1] for similar considerations).

3. Bose randomized occupancy and order statistics

In practice, it is convenient to order the energy levels; by doing so, dependence is naturally introduced. Let then $\mathbf{E}_{(n)} := (E_{(1),n}, \ldots, E_{(n),n})$ be the order statistics of $\mathbf{E}_n$, with $E_{(1),n} < \cdots < E_{(n),n}$. As is well known that the joint probability density of this vector is

$$f_{E_{(1),n}, \ldots, E_{(n),n}}(\epsilon_1, \ldots, \epsilon_n) = n! \prod_{m=1}^n f_1(\epsilon_m) \cdot \mathbf{I}_{\{\epsilon_1 < \cdots < \epsilon_n\}}. \tag{3.1}$$

Thermodynamics under ranked energy levels is the same as the one under unranked ones. This is basically because for symmetric in the arguments functionals $\varphi(\mathbf{E}_n) \overset{\text{a.s.}}{=} \varphi(\mathbf{E}_{(n)})$; conditionally on $\mathbf{E}_n$, thermodynamic functions involved separable and additive functionals of the form $\varphi(\mathbf{E}_n) = \sum_1^n g(E_m)$ for different g. However, $\sum k_m E_m \overset{\text{a.s.}}{\neq} \sum k_m E_{(m),n}$ and this is why sampling under $\mathbf{E}_n$ does not reduce to sampling under $\mathbf{E}_{(n)}$.

For instance, given $\mathbf{E}_{(n)}$, the joint probability of occupancies $K_{(n),\alpha}(m)$ of $E_{(m),n}$; $m = 1, \ldots, n$ now reads

$$\mathbb{P}_{\mathbf{E}_{(n)}}\left(K_{(n),\alpha}(m) = k_m; m = 1, \ldots, n\right) = \frac{1}{Z_{\mathbf{E}_{(n)}}(\alpha, \beta)}\prod_{m=1}^n \mathrm{e}^{-(\alpha+\beta E_{(m),n})k_m}$$

$$Z_{\mathbf{E}_{(n)}}(\alpha, \beta) = \prod_{m=1}^n \frac{1}{1 - \mathrm{e}^{-(\alpha+\beta E_{(m),n})}}.$$

Exchangeability is lost; averaging over $\mathbf{E}_{(n)}$, the unconditional occupancy probability is

$$\mathbb{P}(K_{(n),\alpha}(m) = k_m; m = 1, \ldots, n) := \mathbb{E}\mathbb{P}_{\mathbf{E}_{(n)}}(K_{(n),\alpha}(m) = k_m; m = 1, \ldots, n)$$

$$= \mathbb{E}\left[\prod_{m=1}^n \mathrm{e}^{-(\alpha+\beta E_{(m),n})k_m}(1 - \mathrm{e}^{-(\alpha+\beta E_{(m),n})})\right]$$

$$= n! \int_{\epsilon_1 < \cdots < \epsilon_n} \prod_{m=1}^n \mathrm{e}^{-(\alpha+\beta\epsilon_m)k_m}(1 - \mathrm{e}^{-(\alpha+\beta\epsilon_m)})f_1(\epsilon_m)\,\mathrm{d}\epsilon_m.$$

The occupation numbers of state m with energy $E_{(m),n}$, namely $\mathbb{E}\left(K_{(n),\alpha}(m)\right)$, present some interest in practice. Here for instance, we get a randomized version of Planck statistics

$$\mathbb{E}(K_{(n),\alpha}(m)) = \sum_{k_m \geqslant 1} \mathbb{P}(K_{(n),\alpha}(m) \geqslant k_m) = \mathbb{E}\left(\frac{1}{e^{\alpha+\beta E_{(m),n}} - 1}\right),$$

where, with F_1 the distribution function of E_1 ($\overline{F}_1 := 1 - F_1$), $E_{(m),n} > 0$ has density

$$\frac{n!}{(m-1)!(n-m)!} f_1(\epsilon) F_1(\epsilon)^{m-1} \overline{F}_1(\epsilon)^{n-m}.$$

Let us investigate the full ground-state occupancy distribution in more details.

Lemma 1. *The grand canonical distribution of ground-state occupancy is*

$$\mathbb{P}(K_{(n),\alpha}(1) \geqslant k_1) = e^{-\alpha k_1}\left(1 - \beta \int_0^\infty \left(1 - F_1\left(\frac{x}{k_1}\right)\right)^n e^{-\beta x}\, dx\right). \tag{3.2}$$

Proof. With $\mathbf{E}_{(n)\backslash 1} := (E_{(2),n}, \ldots, E_{(n),n})$, $\phi_{(1)}(\beta) := \mathbb{E}\, e^{-\beta E_{(1),n}}$ and $k_1 \in \mathbb{N}_0$, the ground state $E_{(1),n}$ occupancy is

$$\mathbb{P}(K_{(n),\alpha}(1) = k_1) = \mathbb{E}\left[\frac{e^{-(\alpha+\beta E_{(1),n})k_1} Z_{\mathbf{E}_{(n)\backslash 1}}(\alpha, \beta)}{Z_{\mathbf{E}_{(n)}}(\alpha, \beta)}\right]$$

$$= \mathbb{E}[e^{-(\alpha+\beta E_{(1),n})k_1}(1 - e^{-(\alpha+\beta E_{(1),n})})]$$

$$= e^{-\alpha k_1}[\phi_{(1)}(\beta k_1) - e^{-\alpha}\phi_{(1)}(\beta(k_1 + 1))].$$

With $\overline{F}_1(\epsilon_1) := \mathbb{P}(E_1 > \epsilon_1)$, $\overline{F}_{(1)}(\epsilon_1) := \mathbb{P}(E_{(1),n} > \epsilon_1)$, recalling $\overline{F}_{(1)}(\epsilon_1) = \overline{F}_1(\epsilon_1)^n$, we get

$$\phi_{(1)}(\beta) := \mathbb{E}[e^{-\beta E_{(1),n}}] = 1 - \beta \int_0^\infty \overline{F}_1(\epsilon_1)^n\, e^{-\beta \epsilon_1}\, d\epsilon_1.$$

Therefore equation (3.2) holds. $\qquad\square$

The last formula allows to extract some limiting information on the ground-state occupancy at critical point.

A limit law at criticality from example (i). Assume $F_1(\epsilon) \sim_{\epsilon\downarrow 0} C \cdot \epsilon^\theta$ with $\theta > 1$, as in example (i). Approaching critical point, $\alpha \downarrow 0$, $\rho \uparrow \rho_c(\beta) = H_\beta(1)$ and so, with $k_{n,c} := n\rho_c(\beta)$ the critical number of particles in the system

$$\mathbb{P}\left(K_{(n),0}(1) \geqslant k_{n,c}^{1/\theta} z\right) = 1 - \beta \int_0^\infty \left(1 - F_1\left(\frac{x}{k_{n,c}^{1/\theta} z}\right)\right)^n e^{-\beta x}\, dx$$

$$\sim_{n\uparrow\infty} 1 - \beta \int_0^\infty e^{-n F_1\left(\frac{x}{k_{n,c}^{1/\theta} z}\right)} e^{-\beta x}\, dx$$

$$\sim_{n\uparrow\infty} 1 - \beta \int_0^\infty e^{-\frac{C}{\rho_c}\left(\frac{z}{x}\right)^{-\theta}} e^{-\beta x}\, dx =: \overline{G}(z),$$

where it can be checked that $\overline{G}(z), z > 0$, as a scale exponential mixture of the Fréchet distribution $e^{-\frac{C}{\rho_c} z^{-\theta}}$, is a complementary distribution function of some random variable $Z > 0$. Thus, we obtain

Corollary 1. *Fix $\beta > 0$ and assume $F_1(\epsilon) \sim_{\epsilon\downarrow 0} C \cdot \epsilon^\theta$. Then, with $k_{n,c} := n\rho_c(\beta)$ the critical number of particles in the system at temperature β^{-1}, at critical point*

$$\frac{K_{(n),0}(1)}{k_{n,c}^{1/\theta}} \xrightarrow{d}_{n\uparrow\infty} Z. \tag{3.3}$$

Ground-state occupancy grows like $k_{n,c}^{1/\theta}$ which is close to $n\rho_c$ only as θ approaches 1 from above.

Some remarks on the condensed phase.   Fix $\beta > 0$ and assume condensation takes place. In the super-critical condensed phase $\rho > \rho_c(\beta)$, there is no α such that Bose condition holds. Therefore, the grand canonical approach is useless to describe the condensed phase and, with $\kappa_n = \lfloor n\rho \rfloor$, one needs to use back the canonical ensemble description $K_{n,\kappa_n}(m), m = 1, \ldots, n$ of the occupancy problem. We have the following result:

Proposition 4. *Assume $\mathbb{E}\big(1/E_1^2\big) < \infty$, so that $\mathbb{E}(1/E_1) < \infty$ and condensation takes place. Let $\rho > \rho_c$ be the density of particles in the condensed phase. Under this assumption, almost surely*

$$\frac{1}{n} \sum_{m=2}^{n} \frac{\mathrm{e}^{-\beta \widetilde{E}_{(m),n}}}{1 - \mathrm{e}^{-\beta \widetilde{E}_{(m),n}}} \underset{n\uparrow\infty}{\to} \rho_c = \rho_c(\beta) = \mathbb{E}\left[\frac{\mathrm{e}^{-\beta E_1}}{1 - \mathrm{e}^{-\beta E_1}} \right] < \infty,$$

so that a proportion $n(\rho - \rho_c)$ of particles goes to ground state, while the residual $n\rho_c$ is scattered on all remaining energy levels.

Sketch of proof. First, the minimal value $E_{(1),n}$ from $\mathbf{E}_n$ goes to 0 almost surely. Under fairly general assumptions indeed (see [8]), the almost sure behavior of partial minima reads

$$\liminf_{n\uparrow\infty} \frac{E_{(1),n}}{\epsilon_n} = 1 \text{ a.s.},$$

where $\epsilon_n \downarrow 0$ is defined by $F_1(\epsilon_n) = n^{-1}$. We shall say $E_{(1),n} \sim \epsilon_n$.
   Let now $g(x) := (\mathrm{e}^{\beta x} - 1)^{-1}$. If there is condensation (or if $\mathbb{E}(1/E_1) < \infty$), then as $n \uparrow \infty$

$$\frac{1}{n} g(E_{(1),n}) \to 0$$

because, in the worst case $E_{(1),n} \sim n^{-1/\theta}, \theta > 1$ and then $\frac{1}{n} g(E_{(1),n}) \sim n^{1/\theta - 1} \to 0$.
   Recall now from equation (2.5) that when $k = \kappa_n$ is large, the joint law of occupancy numbers $K_{(n),\kappa_n}(m); m = 2, \ldots, n$ for all states but ground state is (up to a constant going exponentially fast to 0 with $\kappa_n \uparrow \infty$) close to the one of a sequence of independent geometrically distributed random variables with success probabilities $\mathrm{e}^{-\beta \widetilde{E}_{(m),n}}$, where $\widetilde{E}_{(m),n} := E_{(m),n} - E_{(1),n}$ (see the proof of proposition 1). Let us therefore investigate the large n behavior of the sample mean number of particles

$$\frac{1}{n} \sum_{m=2}^{n} \frac{\mathrm{e}^{-\beta \widetilde{E}_{(m),n}}}{1 - \mathrm{e}^{-\beta \widetilde{E}_{(m),n}}} =: \frac{1}{n} \sum_{m=2}^{n} g(\widetilde{E}_{(m),n})$$

on all states but ground state. Recalling $E_{(1),n}$ goes to 0 almost surely, we have

$$g(\widetilde{E}_{(m),n}) \sim \frac{1}{\mathrm{e}^{\beta E_{(m),n}}(1 - \beta E_{(1),n}) - 1}$$

$$\sim \frac{1}{\mathrm{e}^{\beta E_{(m),n}} - 1} \left(1 + \frac{\beta E_{(1),n}}{1 - \mathrm{e}^{-\beta E_{(m),n}}} \right).$$

Therefore,

$$\frac{1}{n} \sum_{m=2}^{n} g(\widetilde{E}_{(m),n}) \sim \frac{1}{n} \sum_{m=2}^{n} g(E_{(m),n}) + \frac{\beta E_{(1),n}}{n} \sum_{m=2}^{n} \frac{\mathrm{e}^{-\beta E_{(m),n}}}{(1 - \mathrm{e}^{-\beta E_{(m),n}})^2}.$$

Under the additional assumption $\mathbb{E}(1/E_1^2) < \infty$, the corrective term goes to 0 when $n \uparrow \infty$ (with $E_{(1),n} \downarrow 0$). Indeed,

$$\frac{\beta E_{(1),n}}{n} \sum_{m=2}^{n} \frac{e^{-\beta E_{(m),n}}}{(1 - e^{-\beta E_{(m),n}})^2} = \beta E_{(1),n} \left(\frac{1}{n} \sum_{m=1}^{n} \frac{e^{-\beta E_{(m),n}}}{(1 - e^{-\beta E_{(m),n}})^2} - \frac{1}{n} \frac{e^{-\beta E_{(1),n}}}{(1 - e^{-\beta E_{(1),n}})^2} \right)$$

and, by strong law of large numbers,

$$\frac{1}{n} \sum_{m=1}^{n} \frac{e^{-\beta E_{(m),n}}}{(1 - e^{-\beta E_{(m),n}})^2} \to \sigma^2 = \mathbb{E}\left(\frac{e^{-\beta E_1}}{(1 - e^{-\beta E_1})^2} \right),$$

which is a finite variance if $\mathbb{E}\left(1/E_1^2\right) < \infty$. Next,

$$\frac{\beta E_{(1),n}}{n} \frac{e^{-\beta E_{(1),n}}}{(1 - e^{-\beta E_{(1),n}})^2} \sim \frac{1}{n \beta E_{(1),n}} \to 0.$$

Therefore, $\frac{1}{n} \sum_{m=2}^{n} g(\widetilde{E}_{(m),n}) \sim \frac{1}{n} \sum_{m=1}^{n} g(E_{(m),n}) = \frac{1}{n} \sum_{m=1}^{n} g(E_m)$ and by strong law of large numbers

$$\frac{1}{n} \sum_{m=2}^{n} g(\widetilde{E}_{(m),n}) \underset{n \uparrow \infty}{\to} \rho_c = \rho_c(\beta) = \mathbb{E}[g(E_1)]. \qquad \square$$

In a random scheme of allocating particles, it is said that a giant component appears if, as $k, n \to \infty$ while the ratio $\rho = k/n$ is fixed to $\rho > \rho_c$, the maximum $K_{(n),k}(1)$ of the random variables $K_{n,k}(m), m = 1, \ldots, n$, with probability tending to 1 is of order n and the second-order statistic has a limit distribution with normalizing constant of order less than n. In the condensed phase, $K_{(n),k}(1)$ (the number of particles attached to ground state) is of order of magnitude $n(\rho - \rho_c)$. Let us investigate $K_{(n),k}(2)$ (the number of particles on state just above ground state) and check this from an example.

First, $\mathbb{P}(E_{(1),n} > \epsilon_1) = \overline{F}(\epsilon_1)^n$ and, for $m = 1, \ldots, n - 1$ we have the Markov property

$$\mathbb{P}(E_{(m+1),n} > \epsilon_{m+1} \mid E_{(m),n} = \epsilon_m, \ldots, E_{(1),n} = \epsilon_1)$$

$$= \mathbb{P}(E_{(m+1),n} > \epsilon_{m+1} \mid E_{(m),n} = \epsilon_m) = \left(\frac{\overline{F}_1(\epsilon_{m+1})}{\overline{F}_1(\epsilon_m)} \right)^{n-m}; \qquad \epsilon_{m+1} > \epsilon_m$$

involving the cumulative probability distribution function of the minimum of $n - m$ iid observations drawn from $\overline{F}_1$ truncated at ϵ_m. If $m = 1$, with $\epsilon_2 > \epsilon_1$

$$\mathbb{P}(E_{(2),n} > \epsilon_2 \mid E_{(1),n} = \epsilon_1) = \left(\frac{\overline{F}_1(\epsilon_2)}{\overline{F}_1(\epsilon_1)} \right)^{n-1}$$

and so with $\epsilon > 0$

$$\mathbb{P}(\widetilde{E}_{(2),n} > \epsilon \mid E_{(1),n} = \epsilon_1) = \left(\frac{\overline{F}_1(\epsilon_1 + \epsilon)}{\overline{F}_1(\epsilon_1)} \right)^{n-1}$$

$$= \left(1 - \frac{\overline{F}_1(\epsilon_1) - \overline{F}_1(\epsilon_1 + \epsilon)}{\overline{F}_1(\epsilon_1)} \right)^{n-1}.$$

From this latter expression, assuming n is large and $F_1(\epsilon) \sim_{\epsilon \downarrow 0} C \cdot \epsilon^\theta, \theta > 1$, one can indeed estimate $E_{(1),n}$ and $\widetilde{E}_{(2),n}$ to be both of order of magnitude $n^{-1/\theta}$. Therefore, the average occupancy of state $E_{(2),n}$ is $g(\widetilde{E}_{(2),n}) \sim n^{1/\theta}$ which is of order less than n.

4. Sampling formulae

Sampling formula. Let $p \in \{1, \ldots, n\}$ and $1 \leqslant m_1 < \cdots < m_p \leqslant n$ be an increasing subsequence of $\{1, \ldots, n\}$. Let $M_1, \ldots, M_p$ be the labels of occupied energy levels and $P_{n,\alpha}$

denote the number of such occupied states. With $k_q \in \mathbb{N} := \{1, 2, \ldots\}$, a realization of occupancies $K_{(n),\alpha}(m_q)$, we clearly have

Proposition 5 (sampling formula)**.**

$$\mathbb{P}_{\mathbf{E}_{(n)}}(M_q = m_q, K_{(n),\alpha}(m_q) = k_q; q = 1, \ldots, p; P_{n,\alpha} = p)$$

$$= \prod_{q=1}^{p} [e^{-(\alpha+\beta E_{(m_q),n})k_q}(1 - e^{-(\alpha+\beta E_{(m_q),n})})] \prod_{q\neq 1,\ldots,p} [1 - e^{-(\alpha+\beta E_{(m_q),n})}].$$

Averaging over $\mathbf{E}_{(n)}$ *with joint law given by equation (3.1), gives the unconditional probability of the event* $M_q = m_q, K_{(n),\alpha}(m_q) = k_q; q = 1, \ldots, p; P_{n,\alpha} = p$.

Summing next the above conditional probability over $k_q; q = 1, \ldots, p$, we get

$$\mathbb{P}_{\mathbf{E}_{(n)}}(M_q = m_q; q = 1, \ldots, p; P_{n,\alpha} = p) = \prod_{q\neq\{1,\ldots,p\}} [1 - e^{-(\alpha+\beta E_{(m_q),n})}]$$

$$\times \sum_{k_1,\ldots,k_p \geqslant 1} \prod_{q=1}^{p} [e^{-(\alpha+\beta E_{(m_q),n})k_q}(1 - e^{-(\alpha+\beta E_{(m_q),n})})]$$

$$= \prod_{q\neq\{1,\ldots,p\}} [1 - e^{-(\alpha+\beta E_{(m_q),n})}] \prod_{q=1}^{p} e^{-(\alpha+\beta E_{(m_q),n})}. \tag{4.1}$$

Averaging over $\mathbf{E}_{(n)}$, using the joint law of $\mathbf{E}_{(n)}$, we get the unconditional probability $\mathbb{P}(M_q = m_q; q = 1, \ldots, p; P_{n,\alpha} = p)$ that only $p \leqslant n$ of the ordered energy levels with labels m_q are occupied. In particular,

$$\mathbb{E}\left\{\prod_{q=2}^{n} [1 - e^{-(\alpha+\beta E_{(q),n})}]e^{-(\alpha+\beta E_{(1),n})}\right\} \tag{4.2}$$

is the unconditional probability that only ground state $E_{(1),n}$ is occupied.

Summing over the $\binom{n}{p}$ sequences $m_1 < \cdots < m_p$, with $p \in \{0, \ldots, n\}$, we obtain a binomial distribution for the number of occupied energy states

$$\mathbb{P}(P_{n,\alpha} = p) = \binom{n}{p}[e^{-\alpha}\phi_1(\beta)]^p(1 - e^{-\alpha}\phi_1(\beta))^{n-p}.$$

Note that $\mathbb{P}(P_{n,\alpha} = 0) = (1 - e^{-\alpha}\phi_1(\beta))^n$ which is the Gibbs probability that there is no particle in the system: $K_{n,\alpha} = 0$.

Corollary 2. *Fix* $\beta > 0$ *and let* $k_{n,c} := n\rho_c(\beta)$ *be the critical number of particles in the system. Then, as* $n \uparrow \infty, \alpha \downarrow 0$, *approaching critical point from the fluid phase*

$$\frac{P_{n,\alpha}}{n\rho} \sim \frac{P_{n,0}}{k_{n,c}} \xrightarrow{d} \frac{\phi_1(\beta)}{\rho_c(\beta)} < 1. \tag{4.3}$$

Proof. With $\lambda \geqslant 0$, the LST of $\frac{P_{n,0}}{k_{n,c}}$ reads

$$\mathbb{E}(e^{-\lambda\frac{P_{n,0}}{k_{n,c}}}) = \left(1 - \left(1 - e^{-\frac{\lambda}{k_{n,c}}}\right)\phi_1(\beta)\right)^n \sim e^{-\frac{\lambda\phi_1(\beta)}{\rho_c(\beta)}}.$$

From the state equation, $\rho_c(\beta) = H_\beta(1) = \sum_{i \geqslant 1} \phi_1(\beta i) > \phi_1(\beta)$. At critical point, the degenerate (non-random) fraction $\frac{\phi_1(\beta)}{\rho_c(\beta)}$ of states is occupied. $\square$

Remark (a non-degenerate low temperature $*$-limit at $\nu < \infty$). The above results suggest that the following 'star' limit should also be worth considered: assume $n \uparrow \infty$, $\beta \uparrow \infty$ (so that $\phi_1(\beta) \downarrow 0$) in such a way that $n\phi_1(\beta) = \gamma > 0$.

Then

$$P_{n,\alpha} \to_* P_\alpha \overset{d}{\sim} \text{Poisson}(\gamma\, e^{-\alpha}).$$

Recalling

$$\nu = nzh_\beta(z) \qquad \text{where} \qquad zh_\beta(z) = \sum_{i \geqslant 1} z^i \phi_1(\beta i),$$

in case $r_2(\beta) := \phi_1(2\beta)/\phi_1(\beta) \to_{\beta \uparrow \infty} 0$, to the dominant term in β, $zh_\beta(z) \sim_* z\phi_1(\beta)$. Thus, $\nu \sim_* nz\phi_1(\beta) \sim_* \gamma z = \gamma\, e^{-\alpha}$. Thus, in this asymptotic, $P_\alpha \overset{d}{\sim} \text{Poisson}(\nu)$ where ν is the finite expected number of particles in the system. Note that in such limiting situations, density $\rho = \nu/n \to_* 0$: in the *-limit, both temperature and density vanish simultaneously.

Asymptotics of the statistics of the number of energy states with prescribed amount of particles. The understanding of the number of occupied (and therefore also of unoccupied) states is part of the broader problem of the number of states with prescribed amount of particles. Let therefore

$$A_{n,\alpha}(i) := \sum_{m=1}^{n} \mathbf{I}_{\{K_{n,\alpha}(m)=i\}}; i \geqslant 0 \tag{4.4}$$

be the number of energy states with exactly i particles. Clearly, $A_{n,\alpha}(0) = n - P_{n,\alpha}$ is the number of free states and $\sum_{i \geqslant 0} A_{n,\alpha}(i) = n$. The case $i = 1$ ($i = 2$) corresponds to singleton (doubleton) states. Recall first

$$\mathbb{P}(K_{n,\alpha}(m) = k_m; m = 1, \ldots, n) = \prod_{m=1}^{n} \mathbb{P}(\xi_{m,\alpha} = k_m),$$

where $(\xi_{1,\alpha}, \ldots, \xi_{n,\alpha})$ is an iid sequence with $\mathbb{P}(\xi_{1,\alpha} \geqslant k_1) = e^{-\alpha k_1}\phi_1(\beta k_1)$. From this, for all sequences $(a_i; i \geqslant 0)$ satisfying the constraint $\sum_{i \geqslant 0} a_i = n$, we easily get

$$\mathbb{P}(A_{n,\alpha}(1) = a_1, \ldots, A_{n,\alpha}(i) = a_i, \ldots) = n! \cdot \prod_{i=0}^{\sum_{l \geqslant 1} l a_l} \left\{ \frac{\mathbb{P}(\xi_{1,\alpha} = i)^{a_i}}{a_i!} \right\}.$$

When $i \geqslant 1$, we have $A_{n,\alpha}(i) = \sum_{m=1}^{n} \mathbf{I}_{\{K_{n,\alpha}(m)=i\}}$, so that

$$\mathbb{E}(A_{n,\alpha}(i)) = \sum_{m=1}^{n} \mathbb{P}(K_{n,\alpha}(m) = i) = \sum_{m=1}^{n} \mathbb{P}(\xi_{m,\alpha} = i)$$
$$= n\mathbb{P}(\xi_{1,\alpha} = i) = n\, e^{-\alpha i}[\phi_1(\beta i) - e^{-\alpha}\phi_1(\beta(i+1))].$$

Thus, consistently,

$$\mathbb{E}(K_{n,\alpha}) = \sum_{i \geqslant 1} i\mathbb{E}(A_{n,\alpha}(i)) = n \sum_{i \geqslant 1} i\, e^{-\alpha i}[\phi_1(\beta i) - e^{-\alpha}\phi_1(\beta(i+1))]$$
$$= n \sum_{i \geqslant 1} e^{-\alpha i}\phi_1(\beta i) = n\mathbb{E}((e^{\alpha+\beta E_1} - 1)^{-1}).$$

Therefore, we obtain

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Proposition 6.

(i) *For all sequences* $(a_i; i \geqslant 0)$ *satisfying* $\sum_{i \geqslant 0} a_i = n$:

$$\mathbb{P}(A_{n,\alpha}(1) = a_1, \ldots, A_{n,\alpha}(i) = a_i, \ldots) = n! \cdot \prod_{i=0}^{\sum_{l \geqslant 1} l a_l} \left\{ \frac{\mathbb{P}(\xi_{1,\alpha} = i)^{a_i}}{a_i!} \right\}. \tag{4.5}$$

(ii) *Assume* $n \uparrow \infty, \beta \uparrow \infty$ *in such a way that* $n\phi_1(\beta) := \gamma > 0$. *In this weak *-limit,* $(A_{n,\alpha}(i); i \geqslant 1)$ *converges to a sequence of independent random elements with law given by* $A_\alpha(i) \overset{d}{\sim} Poisson(\lambda_i)$, $i \geqslant 1$, *where*

$$\lambda_i := \gamma \, \mathrm{e}^{-\alpha i}[r_i - \mathrm{e}^{-\alpha} r_{i+1}] \tag{4.6}$$

and

$$r_i := \lim_{\beta \uparrow \infty} r_i(\beta) \qquad with \quad r_i(\beta) := \phi_1(\beta i)/\phi_1(\beta). \tag{4.7}$$

(iii) *In the weak *-limit, the number of visited states* $P_{n,\alpha}$ *converges to* $P_\alpha \overset{d}{\sim} Poisson(\gamma \, \mathrm{e}^{-\alpha})$.

Proof. It remains to prove (ii) and (iii).

Observing $\mathbb{P}(\xi_{1,\alpha} = 0) = 1 - \mathrm{e}^{-\alpha} \phi_1(\beta)$ and $a_0 = n - \sum_{i \geqslant 1} a_i$, we have

$$\mathbb{P}(A_{n,\alpha}(1) = a_1, \ldots, A_{n,\alpha}(i) = a_i, \ldots) = \frac{n!}{\left(n - \sum_{i \geqslant 1} a_i\right)!} (1 - \mathrm{e}^{-\alpha} \phi_1(\beta))^n$$

$$\times \prod_{i=1}^{\sum_{l \geqslant 1} l a_l} \left\{ \frac{(\mathbb{P}(\xi_{1,\alpha} = i)/(1 - \mathrm{e}^{-\alpha} \phi_1(\beta)))^{a_i}}{a_i!} \right\}$$

$$\sim {}_* n^{\sum_{i \geqslant 1} a_i} \, \mathrm{e}^{-\gamma \, \mathrm{e}^{-\alpha}} \prod_{i \geqslant 1} \left\{ \frac{\phi_1(\beta)^{a_i} (\mathrm{e}^{-\alpha i} r_i(\beta) - \mathrm{e}^{-\alpha(i+1)} r_{i+1}(\beta))^{a_i}}{a_i!} \right\}$$

$$\sim {}_* \prod_{i \geqslant 1} \frac{\lambda_i^{a_i} \, \mathrm{e}^{-\lambda_i}}{a_i!}.$$

In the last equivalence, we used $\gamma \, \mathrm{e}^{-\alpha} = \sum_{i \geqslant 1} \lambda_i$. The law of $(A_{n,\alpha}(i); i \geqslant 1)$ therefore converges to the infinite product of Poisson factors

$$\mathbb{P}^*(A_\alpha(1) = a_1, \ldots, A_\alpha(i) = a_i, \ldots) = \prod_{i \geqslant 1} \frac{\lambda_i^{a_i} \, \mathrm{e}^{-\lambda_i}}{a_i!}$$

with the announced finite-dimensional distributions in (ii). The constants $\nu := \mathbb{E}^*(K_\alpha), \alpha > 0$ and $\gamma > 0$ are now related by

$$\nu = \sum_{i \geqslant 1} i \lambda_i = \gamma \sum_{i \geqslant 1} \mathrm{e}^{-\alpha i} r_i.$$

To prove (iii) again, observe $P_\alpha = \sum_{i \geqslant 1} A_\alpha(i)$ and use the Poisson decomposition. □

Example.

(i) Assume $r_2(\beta) := \phi_1(2\beta)/\phi_1(\beta) \to_{\beta \uparrow \infty} r_2 = 0$ as in examples (ii) and (iii). Then for all $i \geqslant 2, r_i(\beta) \to r_i = 0$ and so $\lambda_i \to 0$ $(\beta \uparrow \infty)$. Thus, $A_{n,\alpha}(i) \neq 0$ with some positive probability only when $i = 1$ (singletons) and $\mathbb{P}^*(A_\alpha(1) = a_1) = \frac{\lambda_1^{a_1} \, \mathrm{e}^{-\lambda_1}}{a_1!}$ where $\lambda_1 = \gamma \, \mathrm{e}^{-\alpha} = \nu$. This asymptotic regime is the one of uniques. States whose occupancies cannot exceed 1 are currently obtained in a Fermi–Dirac context. We have no physical interpretation of this statement.

(ii) Assume $r_2(\beta) := \phi_1(2\beta)/\phi_1(\beta) \to_{\beta\uparrow\infty} r_2 = 2^{-\theta}$ as in the regularly varying example (i). Then, $r_i(\beta) \sim_{\beta\uparrow\infty} i^{-\theta}$ and $\lambda_i \sim_{\beta\uparrow\infty} \gamma(1 - e^{-\alpha}) e^{-\alpha i} i^{-\theta}$; configurations with $A_{n,\alpha}(i) > 0$ exist for all $i \geqslant 1$.

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References

[1] Beghian L E and Sheldon E 2001 The random-variable canonical distribution *J. Phys. A: Math. Gen.* **34** 2913–21
[2] Bingham N H, Goldie C M and Teugels J L 1987 *Regular Variation (Encyclopedia of Mathematics and its Applications* vol 27*)* (Cambridge: Cambridge University Press) 491 pp
[3] Bose S N 1924 Plancks gesetz und Lichtquantenhypothese *Z. Phys.* **26** 178–81
[4] Brézin E 1993 *Physique Statistique* (Cours édité par l' Ecole Polytechnique)
[5] Derrida B 1981 Random-energy model: an exactly solvable model of disordered systems *Phys. Rev.* B **24** 2613–26
[6] Derrida B 1980 The random energy model: common trends in particle and condensed matter physics *Proc. Winter Adv. Study Inst., Les Houches*
[7] Derrida B 1980 Random-energy model: limit of a family of disordered models *Phys. Rev. Lett.* **45** 79–82
[8] Embrechts P, Klüppelberg C and Mikosh T 1997 *Modelling Extremal Events (Applications of Mathematics* vol 33*)* (Berlin: Springer)
[9] Evans M R 1996 Bose–Einstein condensation in disordered exclusion models and relation to traffic flow *Europhys. Lett.* **36** 13–8
[10] Evans M R, Majumdar S N and Zia P R K 2004 Factorized steady states in mass transport models *J. Phys. A: Math. Gen.* **37** L275–80
[11] Evans M R, Majumdar S N and Zia P R K 2006 Canonical analysis of condensation in factorised steady state *J. Stat. Phys.* **123** 357–90
[12] Feller W 1971 *An Introduction to Probability Theory and its Applications* vols 1 and 2, 2nd edn (New York: Wiley)
[13] Holst L 1985 On discrete spacings and the Bose–Einstein distribution *Contributions to Probability and Statistics (Essays in Honour of Gunnar Blom)* ed J Lanke and G Lindgren (Lund) pp 169–77
[14] Johnson N L and Kotz S 1977 *Urn Models and Their Application: An Approach to Modern Discrete Probability Theory (Wiley Series in Probability and Mathematical Statistics)* (New York: Wiley) xiii+402 pp
[15] Kolchin V F 1986 *Random Mappings (Translated from the Russian. With a foreword by S R S Varadhan) (Translation Series in Mathematics and Engineering)* (New York: Optimization Software, Inc.)
[16] Lauwers J, Verbeure A and Zagrebnov V A 2003 Bose–Einstein condensation for homogeneous interacting systems with a one-particle spectral gap *J. Stat. Phys.* **112** 397–420
[17] Lenoble O and Zagrebnov V 2007 Bose–Einstein condensation in the Luttinger-sy model *Markov Process. Relat. Fields* **13** 441–68